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On the Pentastroid.[*]

By R. P. Stephens.

I. *Introduction.*

In an article entitled "On a System of Parastroids," in the July number of the *Annals of Mathematics*, the equations of the curves arising from the Wallace lines were found to be of the form

$$t^n + xt^{n-1} + a_1 t^{n-2} + a_2 t^{n-3} + \ldots . + b_2 t^3 + b_1 t^2 + yt + 1 = 0,$$

where x and a_i have for conjugates y and b_i respectively, and t is a parameter which is limited to the unit circle. In the particular case where $n = 3$, this is the equation of a deltoid, or hypocycloid of three cusps; and where $n = 4$, it is the equation of the parastroid. I propose to discuss the nature of the curve when $n = 5$, but I shall also call attention to a few theorems which are true for the general case. The coordinates used are circular or conjugate, however most of the work will be done by means of mapping from the unit circle.

II. *Mechanical Construction.*[†]

When $n = 5$, the equation given above takes the form

$$t^5 + xt^4 + at^3 + bt^2 + yt + 1 = 0,$$

which may be written

$$xt^3 + y = t^3(-t - a/t) + (-1/t - bt).$$

This second form is obviously the equation of a straight line which always passes through the point

$$x = -t - a/t.$$

But if t is allowed to vary, then this point traces out an ellipse. Whence we see that, if a line be fixed to the generating point of an ellipse and given the proper rotation about this point, its envelope will be the curve in question. If, in Fig. 1, the distance of P from M is μ, and if the rotation of the line l about P is 3/2 that of M about O, then the equation of l is

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[*] A preliminary report of this article was made to the American Mathematical Society, Feb. 23, 1907.

[†] Cf. *Transactions of the American Mathematical Society*, Vol. VII (1906), pp. 207–227.

$$t^5 - xt^4 + \mu t^3 - \alpha\mu t^2 + \alpha yt - \alpha = 0, \tag{1}$$

where α is the clinant of the line l when $t = 1$. The equation of the ellipse generated at the same time is

$$x = t + \mu/t. \tag{2}$$

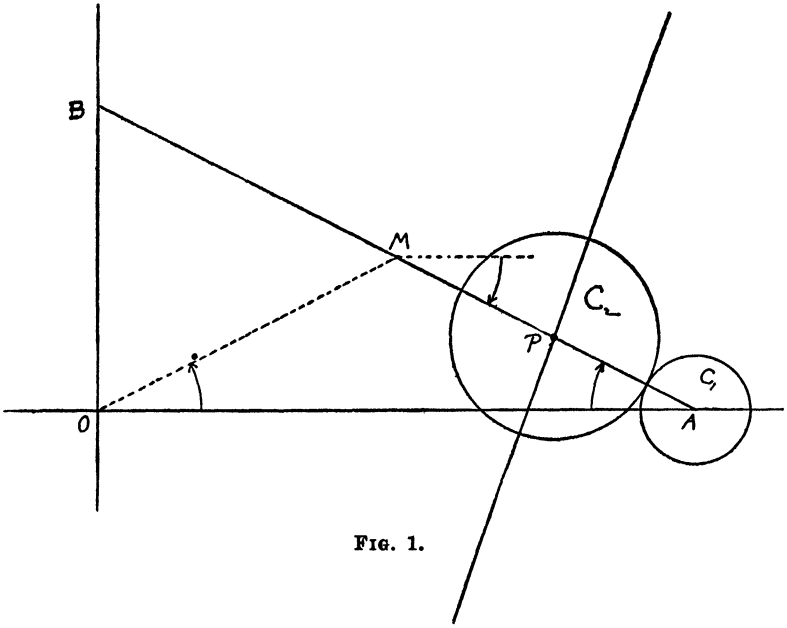

Fig. 1.

In Fig. 2 is a diagram of the instrument as used in the construction of the figures which follow. The gears G_1 and G_2 are to each other as 1 : 2. The first is centered on AB and OA and does not rotate; while G_2, centered on AB, rotates about P.

If a different combination of gears—say G_5, G_2, G, where G_5 and G_2 are to each other as 5 : 2 and G is any connecting gear—be used, the same curve is obtained. In Fig. 1, G_5 is to replace G_1; G_2 remains unchanged; and G is between G_5 and G_2. This combination gives the equation

$$\alpha\mu t^5 - \alpha t^4 y + \alpha t^3 - t^2 + xt - \mu = 0. \tag{3}$$

By means of this double generation, every type of the curve (1), arising from varying μ and α, can be drawn. For example, from (1) it is obvious that, when $\mu = 0$, we have the equation of the hypocycloid of five cusps; and from (3), when $\mu = 0$, we obtain the cardioid, a curve of one cusp. These two are the limiting forms of the curve and we shall see that the curve varies from one to five cusps.

It seems well to give a name to the curve (1). For several reasons the name Pentastroid seems appropriate, and so it will be used uniformly in what follows to designate the curve (1). In the particular case where the curve is regular, that is, when $\mu = 0$, I shall retain the ordinary usage and refer to it as a five-cusped hypocycloid.

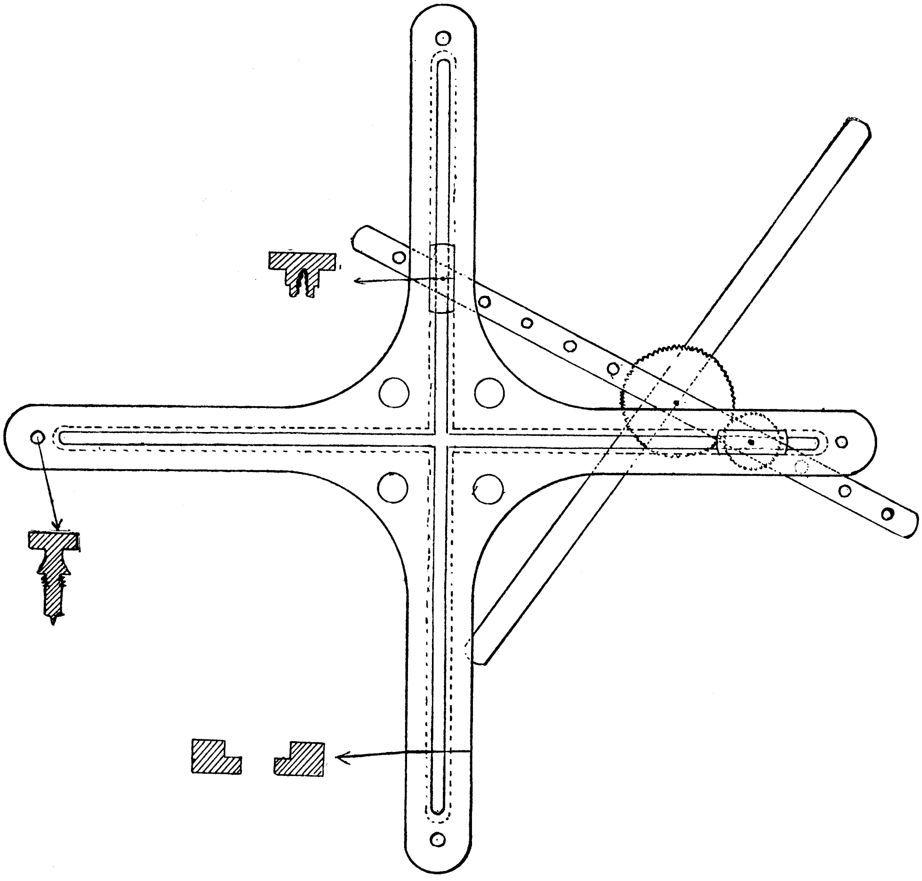

FIG. 2.

III. *The Equation.*

1. Let us consider the equation

$$t^5 - xt^4 + \mu t^3 - \alpha\mu t^2 + \alpha y t - \alpha = 0, \tag{1}$$

which as t varies envelops a curve of *class five,* for from every point there are five tangents to the curve. If (1) be divided by t and then differentiated with respect to t, there results the equation,

$$3x = 4t + 2\mu/t - a\mu/t^2 + a/t^4, \tag{4}$$

the map equation of (1), from which it is seen that the curve is of the *eighth degree.*

2. If, in equation (1), the point x be allowed to move off to infinity, the equation reduces to the form

$$t^3 = ay/x. \tag{5}$$

But y/x is a turn, hence ay/x is a turn; therefore, for x at infinity in any direction, there are three tangents to the curve.

Let the roots of (5) be t_1, ωt_1, $\omega^2 t_1$; then, on substituting these values of t in equation (4) and adding, we obtain

$$\Sigma x_i = 0,$$

where x_i are the points of tangency of these parallel tangents. Therefore,

To a pentastroid there are, in every direction, three parallel tangents, the centroid of whose points of tangency is constant.

In a similar manner, it is proved that

Every curve whose equation has the form

$$t^n + xt^{n-1} + a_1 t^{n-2} + a_2 t^{n-3} + \dots + b_2 t^3 + b_1 t^2 + yt + 1 = 0,$$

has $n - 2$ *parallel tangents in any direction, and the centroid of their points of tangency is constant.*

This fixed point is defined as the center of the curve. For the curve with equation as given the center is the origin.

3. We have seen that the map equation of the basic ellipse is

$$x = t + \mu/t. \tag{2}$$

The clinant* of the tangent to this ellipse is

$$dx/dy = \frac{t^2 - \mu}{\mu t^2 - 1}.$$

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*F. Franklin: Some Applications of Circular Coordinates, AMERICAN JOURNAL OF MATHEMATICS, Vol. XII (1890), p. 162.

The clinant of the tangent to the curve (4) at the point which has the same parameter is

$$dx/dy = \frac{\alpha}{t^3}.$$

These clinants are equal, that is the tangents are parallel, when

$$t^5 - \mu t^3 - \alpha\mu t^2 + \alpha = 0, \tag{6}$$

but this is the condition that the two points of tangency shall coincide. Therefore,

The pentastroid touches its basis ellipse in five points, real or imaginary. (Fig. 3.)

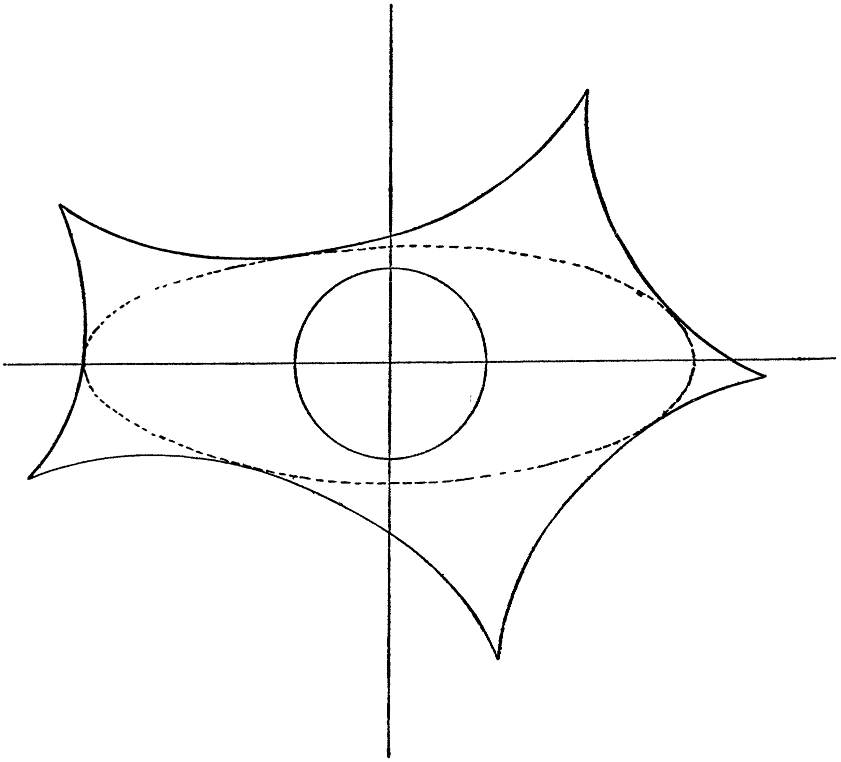

FIG. 3. $\mu = .5$ (nearly).

In the particular case where $\mu = 1$, equation (6) can be solved, and the roots are ± 1, t_1, ωt_1, $\omega^2 t_1$, where t_1 is one of the cube roots of α. Putting these values in the map equation (4), we obtain as the points of tangency

$$x_1 = 2, \qquad x_2 = -2,$$
$$x_3 = t_1 + 1/t_1, \qquad x_4 = \omega t_1 + \omega^2/t_1,$$
$$x = \omega^2 t_1 + \omega/t_1.$$

The ellipse in this case is the segment of the line joining $+2$ and -2. So the pentastroid passes through the points ± 2 for every value of α. The three other points of tangency are on the axis of reals. Hence we see that

When $\mu = 1$, *the pentastroid passes through the points* ± 2, *and the axis of reals is a triple tangent.* (Fig. 4.)

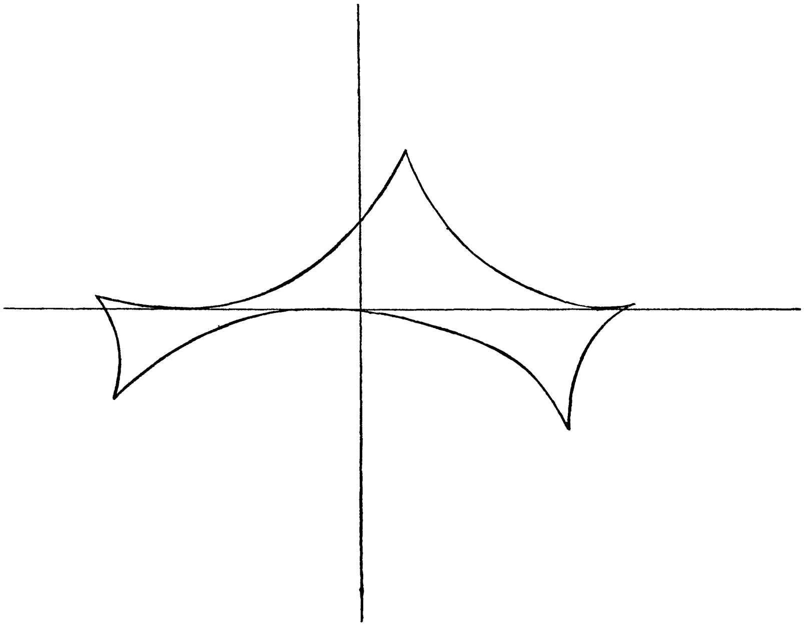

FIG. 4. $\mu = 1$.

4. *Orthoptic Curve.* To the tangent

$$t^5 - xt^4 + \mu t^3 - \alpha\mu t^2 + \alpha y t - \alpha = 0,$$

the tangents obtained by giving t the values $-t$, $-\omega t$, $-\omega^2 t$, will be perpendicular. Substituting the first of these values, we obtain

$$t^5 + xt^4 + \mu t^3 + \alpha\mu t^2 + \alpha y t + \alpha = 0.$$

If this equation be subtracted from the one above, there results

$$x = -\alpha(\mu/t^2 + 1/t^4), \tag{7}$$

which is the map equation of a limaçon. By means of the same method as that used in connection with the basic ellipse, it is easily shown that this limaçon touches the pentastroid in five points whose parameters are given by the equation

$$2t^5 + \mu t^3 + \alpha\mu t^2 + 2\alpha = 0. \tag{8}$$

When $\mu = 2$, this equation can be solved and its roots are $\pm i$, t_1, ωt_1, $\omega^2 t_1$, where t_1 is one of the cube roots of $-\alpha$. Substituting these values in (7), we obtain as the five points of tangency: α (which is counted twice and is therefore a node), $\mu t_1 + 1/t_1$, $\mu\omega t_1 + \omega^2/t_1$, $\mu\omega^2 t_1 + \omega/t_1$. From equation (4) it is seen that $x = \alpha$ is also a node of the pentastroid.

On substituting $-\omega t$ and $-\omega^2 t$ for t in equation (1) and finding the intersections of the resulting equations with (1), we have

$$2x = (1 - \omega)t + \mu(1 - \omega^2)/t + \alpha\mu\omega^2/t^2 + \alpha\omega/t^4 \tag{9}$$

and

$$2x = (1 - \omega^2)t + \mu(1 - \omega)/t + \alpha\mu\omega/t^2 + \alpha\omega^2/t^4,$$

respectively. These two forms are easily shown to represent the same curve, for, when t in the second becomes $-\omega t$, the first results.

This curve also touches the pentastroid in five points. The parameters of these points of tangency are given by the equation

$$(5 + 3\omega)t^5 + \mu(1 + 3\omega^2)t^3 + \alpha\mu(3\omega + 1)t^2 + \alpha(5 + 3\omega^2) = 0. \tag{10}$$

The orthoptic locus of the pentastroid is composed of a limaçon and the curve (9).

5. *Singularities at Infinity.*[*] By comparing equation (1) with the equation $ux + vy = 1$, where u and v are $1/a$ and $1/b$ respectively (a being the reflexion of the origin in the line), the equation of the curve in line coordinates is derived as follows: Equating coefficients, we obtain

$$u = -t^4,$$
$$v = \alpha t,$$
$$w = t^5 + \mu t^3 - \alpha\mu t^2 - \alpha,$$

where w is introduced for the sake of homogeneity. If t be eliminated from these three equations in such a way as to form a homogeneous equation in u, v, and w, the resulting equation is the one required. It is

$$\alpha uvw^3 - [\alpha^2 u^2 (u + \mu v)^3 + v^2 (\mu u + v)^3 + 3uvw (\mu u + v)(u + \mu v)] = 0. \tag{11}$$

This equation can be transformed to Boothian[†] coordinates by the substitution

$$2u = \xi + i\eta, \qquad 2v = \xi - i\eta.$$

Since the coordinates of the line at infinity (0, 0, 1) satisfy equation (11), the curve is tangent to the line at infinity. The points I and J, whose equations are $u = 0$ and $v = 0$ respectively, are such singular points of the curve that the tangent at each has contact of the third order. This tangent is the line at infinity, therefore the line at infinity is a double tangent.

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[*] F. Franklin: *loc. cit.*, pp. 161–190.

[†] Bassett: *Elementary Treatise on Cubic and Quartic Curves*, p. 30.

The pentastroid is a curve which has the line at infinity for a double tangent whose points of tangency are the points I and J, at each of which the contact is of the third order.

From this it follows that *all the foci of the curve are at infinity.*

6. *Cusps.* The curve will have cusps when $dx/dy = 0$, provided the roots of the resulting equation are turns. Thus we find the parameters of cusps are the roots of the equation

$$2t^5 - \mu t^3 + \alpha\mu t^2 - 2\alpha = 0. \tag{12}$$

This equation may have five turns for roots, hence we say in general that *the pentastroid has five cusps, real, coincident, or imaginary.* (Fig. 3.)

On combining equation (1) with equation (12), we obtain

$$3t^5 - xt^4 + \alpha yt - 3\alpha = 0$$

and

$$2xt^3 - 3\mu t^2 + 3\alpha\mu t - 2\alpha y = 0,$$

of which the first is the equation of a regular pentastroid, *i. e.*, a five-cusped hypocycloid, and the second is the equation of a cardioid. Both are concentric with (1). Whence we have the theorem,

The five cusp-tangents of a pentastroid touch a concentric five-cusped hypocycloid, and also a concentric cardioid.

If x and y be written for μ in (12), thus,

$$2t^5 - xt^3 + \alpha yt^2 - 2\alpha = 0, \tag{13}$$

it is obvious that for those values of x (*i. e.* μ) on the axis of reals from which five tangents can be drawn to the regular pentadeltoid (13), equation (1) has five real cusps; that for those values for which (13) has three tangents only, equation (1) has only three real cusps; and that for those values for which (13) has only one tangent, equation (1) has only one real cusp.

However, in the special case where $\alpha = 1$, more definite limits can be stated for μ. Equation (12) now becomes

$$2(t^5 - 1) - \mu t^2 (t - 1) = 0,$$

from which it is seen that $t = 1$ gives a cusp for all values of μ. But since the curve is symmetrical with respect to the axis of reals—when $\alpha = 1$—, if t is a root then $1/t$ is also a root. Suppose then that t_1 and t_2 are roots, then $1/t_1$ and $1/t_2$ are roots also. These relations among the roots make it possible to solve the equation. From the symmetric functions we derive

$$2a = -1 \pm \sqrt{5 + 2\mu}, \tag{14}$$

where $a \equiv t_1 + 1/t_1$.

Evidently a is real and less than, or equal to, 2 in absolute value when t is a turn; hence, in order that a may be real we must have

$$\mu \overset{=}{>} -5/2.$$

The value of a will be less than, or equal to, 2 when

$$1) \qquad -1 + \sqrt{5 + 2\mu} \overset{=}{<} 4,$$

that is, when $\mu \overset{=}{<} 10$; or

$$2) \qquad |-1 - \sqrt{5 + 2\mu}\,| \overset{=}{<} 4,$$

that is, when $\mu \overset{=}{<} 2$. From which we conclude

For all values of μ *such that* $-5/2 \overset{=}{<} \mu \leqq 2$, *the pentastroid for* $\alpha = 1$ *has five cusps; for all values of* μ *such that* $2 < \mu \leqq 10$, *there will be only three real cusps; and for all other values of* μ *there will be only one real cusp.*

The special cases $\mu = -5/2$, 2, 10, when $\alpha = 1$, are interesting. Let us consider first $\mu = -5/2$. Substituting this value of μ in equation (14), and solving for t, we obtain

$$4t = -1 \pm i\sqrt{15},$$

each of which is repeated; that is to say, there are two pairs of coincident cusps. (Fig. 5.)

If $\mu = 10$ when $\alpha = 1$, equation (12) reduces to the form

$$(t - 1)^3 (t^2 + 3t + 1) = 0,$$

of which $t = 1$ is a repeated root. Hence three of the cusps coincide. The other roots are

$$2t = -3 \pm \sqrt{5}.$$

These are not turns and hence do not give real cusps. In Fig. 6 is seen how the cusps tend to disappear as μ increases.

If $\mu = 2$, equation (11) is solvable for all values of α. It takes the form

$$(t^3 + \alpha)(t^2 - 1) = 0,$$

of which the roots are ± 1, t_1, ωt_1, $\omega^2 t_1$, where t_1 is one of the cube roots of $-\alpha$.

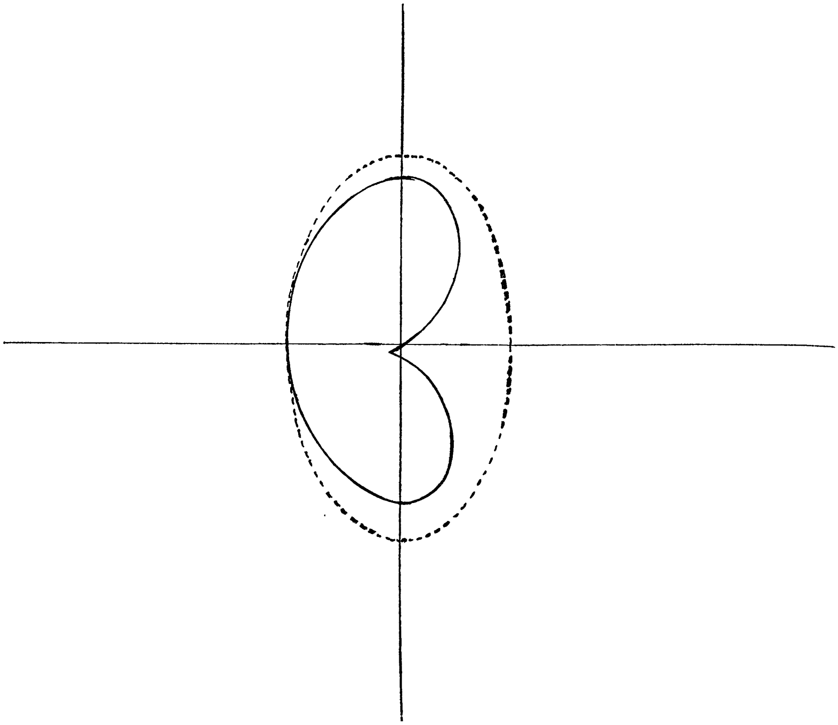

FIG. 5. $\mu = -5/2$.

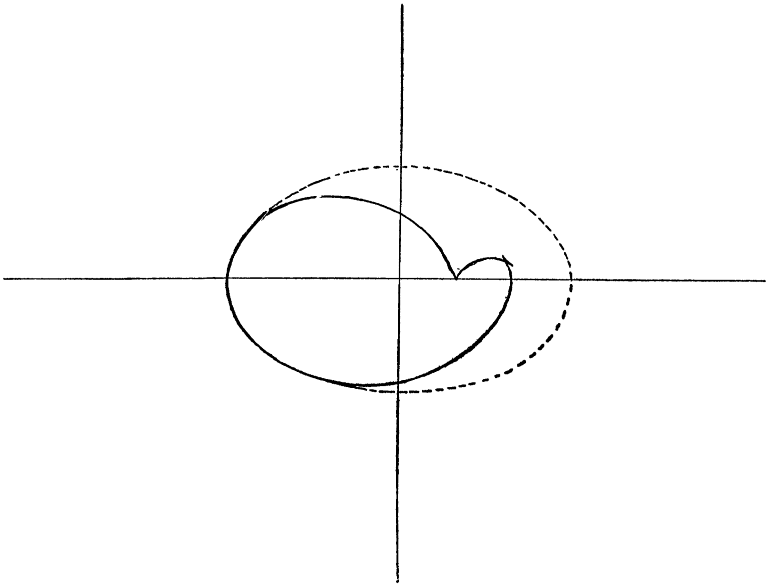

FIG. 6. $\mu = 5$.

Substituting these values of t in equation (4), we find that the five cusps are as follows:

$$
\begin{aligned}
x_1 &= 2\ 2/3 - 1/3\ \alpha,\\
x_2 &= -\ 2\ 2/3 - 1/3\ \alpha,\\
x_3 &= 2t_1 + 1/t_1,\\
x_4 &= 2\omega t_1 + \omega^2/t_1,\\
x_5 &= 2\omega^2 t_1 + \omega/t_1.
\end{aligned}
$$

Now as α varies, the first cusp traces out a circle with radius 1/3 and with center at 2 2/3; the second traces out a circle with the same radius and with center at — 2 2/3; the third, fourth, and fifth trace out the same curve—the basic ellipse of the pentastroid. (Figs. 7, 8, 9.)

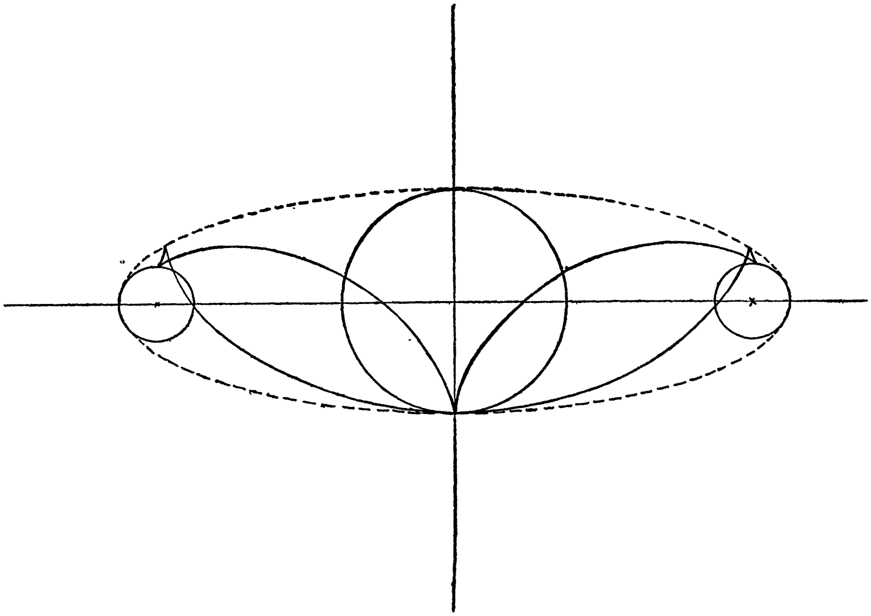

FIG. 7. $\mu = 2.$ $\alpha = i.$

The cusp-tangents at these cusps are:

$$
\begin{aligned}
x - \alpha y &= 3 - 3\alpha,\\
x + \alpha y &= -\ 3 - 3\alpha,\\
x + \ \ y &= 3t_1 + 3/t_1,\\
x + \ \ y &= 3\omega t_1 + 3\omega^2/t_1,\\
x + \ \ y &= 3\omega t_1 + 3\omega^2/t_1,
\end{aligned}
$$

of which the first two are perpendicular lines, passing through the fixed points 3 and — 3, respectively; and the last three are parallel lines perpendicular to the axis of reals.

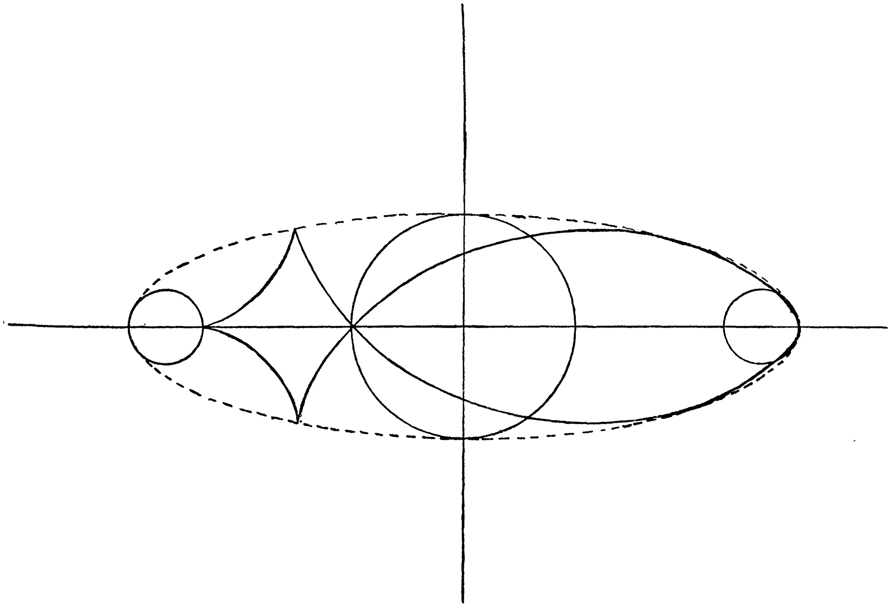

FIG. 8. $\mu=2.$
$a=-1.$

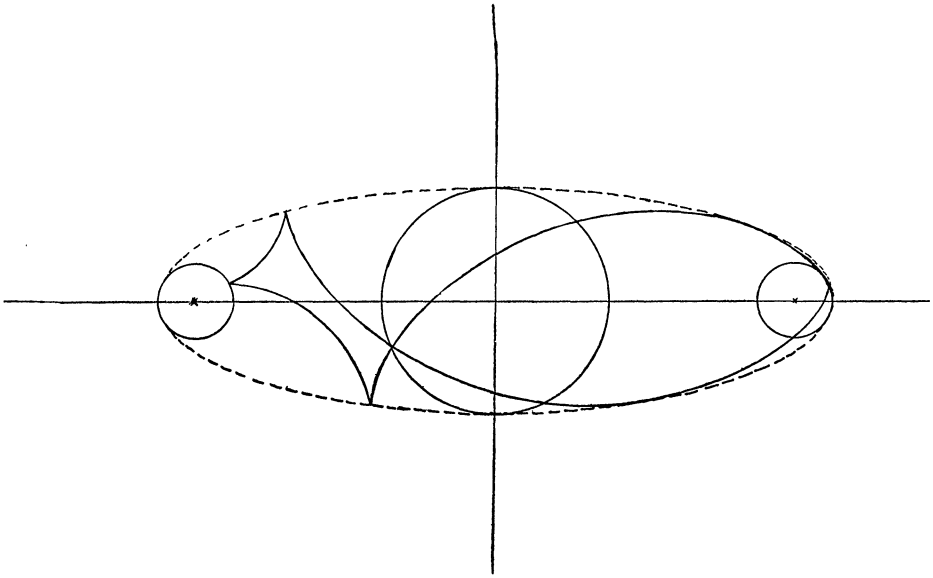

FIG. 9. $\mu=2.$
$a=e^{i\pi/6}$ [Est.].

7. *Miscellaneous Properties.* In § 4 it was seen that when $\mu = 2$, then

$$x = \alpha$$

is a node of the pentastroid. As α varies, this node runs about the unit circle. The tangents at this node are

$$x - i\alpha y = \alpha + i,$$
$$x + i\alpha y = \alpha + i,$$

two lines which are perpendicular; hence, we can say that the curve cuts itself orthogonally at this node. (Figs. 5, 6, 7.)

The equation of the line normal to the tangent

$$t^5 - xt^4 + \mu t^3 - \alpha\mu t^2 + \alpha y t - \alpha = 0$$

at its point of tangency is

$$5t^5 - xt^4 + \mu t^3 + \alpha\mu t^2 - \alpha y t - 5\alpha = 0,$$

a line which envelops another pentastroid concentric with the first. Hence, we have the theorem

The evolute of the pentastroid is a concentric pentastroid.

Since all the foci of the pentastroid are at infinity, it follows immediately from Laguerre's theorem* that *the sum of the reciprocals of the tangents to the curve from any point is zero.*

WESLEYAN UNIVERSITY, *May* 28, 1907.

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* *Oeuvres de Laguerre*, Vol. II, p. 25.